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THE ASYMPTOTIC CHARACTERISTICS OF THE SOLUTIONS OF THE DIFFUSION EQUATION WITH A NON-LINEAR SINK: A RENORMALIZATION GROUP APPROACH†

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A non-linear generalization of the diffusion equation, which describes the mass or heat transfer accompanied with chemical reactions, is used to consider the spreading of an initially localized distribution. The use of a renormalization group method enabled the nature of the solution to be analysed for long times and two characteristics of its asymptotic behaviour to be distinguished. When the dimension of the space is greater than a certain critical value, a state of asymptotic freedom is attained for which the role of non-linearity is small and the evolution of the density distribution is governed by diffusion processes. When the dimension is less than the critical value, the non-linear term remains substantial for long periods of time and a state of incomplete self-similarity of the evolution of the density distribution is established. The exponent of the exponential dependence of the radius of the diffusion spot on time is calculated for this case. The relation between the renormalization group method and perturbation theory and difficulties in substantiating the method when applied to a given problem are discussed.

The interest in the problem of investigating the asymptotic behaviour of particular self-similar solutions of the Cauchy problem for quasilinear parabolic equations is due to the fact that the asymptotic form of the self-similar solution can turn out to be related to a whole class of non-self-similar initial conditions. The permissible self-similar asymptotic forms of the solutions of quasilinear parabolic equations, which are independent of the characteristic dimensional parameters specifying the initial distribution, have been investigated in detail in [1]. However, such asymptotic forms do not exhaust all the possible types of solutions, and other types of limiting solutions may exist which depend on certain integral characteristics of the initial conditions. These solutions correspond to a state of incomplete self-similarity (self-similarity of the second kind [2]), when the dependence on the dimensional parameters, specifying the initial distribution, does not vanish in the limit and has an effect on the exponents of the exponential behaviour in the form of corrections to the values which follow from simple dimensional considerations. The aim of this paper is to demonstrate the possibilities of using the renormalization group method to find solutions which correspond to a state of incomplete self-similarity. Unlike the investigations which have been carried out previously [1, 3], no preliminary assumptions regarding the form of the solution are made when constructing the solution.

1. FORMULATION OF THE PROBLEM

Quasilinear parabolic equations are the basis of the description of a large number of phenomena relating to mechanics, physics, technology, biology, etc. [1, 3, 4]. In particular, in the theory of homogeneous combustion and chemical kinetics, the diffusion equation (for heat or mass transfer)

$$\left[\frac{\partial}{\partial t}+D_0\Delta\right]C(\mathbf{r},t)+\lambda C^{1+2\delta}(\mathbf{r},t)=0 \tag{1.1}$$

is considered with a non-linear source ($\lambda < 0$) or sink ($\lambda > 0$) (the treatment below refers to the case when $\lambda > 0$).

The physical meaning and the dimension of the constant λ depends on the quantity $n = 1 + 2\delta$. In particular, when $\delta = 0$, Eq. (1.1) is linear and the term which is proportional to λ describes the absorption of a substance by

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the medium. When $\delta = 1/2$, the non-linear term describes the change in concentration due to binary-type chemical reactions, coagulation processes accompanying the diffusion of aerosol particles, etc. When $\delta = 3/2$, Eq. (1.1) can be used to analyse heat transfer processes, taking into account radiation losses or radiation heating.

To fix our ideas, we shall subsequently make use of the terminology of chemical kinetics and hence we shall call $C(\mathbf{r}, t)$ the concentration of the substance, D_0 the diffusion coefficient and $n = 1 + 2\delta$ the order of the chemical reaction.

In order to explain the role of non-linear effects and to find the possible asymptotic forms of the solutions of Eq. (1.1), we shall consider the problem of the non-linear diffusive spreading of an initially localized density distribution in an unbounded space, and we shall seek a solution of the Cauchy problem (1.1) with an initial density distribution of delta-like form

$$C(\mathbf{r}, 0) = Q_0\delta(\mathbf{r}) \tag{1.2}$$

This formulation of the problem is equivalent to the addition of an instantaneous source of the form

$$\rho(\mathbf{r}, t) = Q_0\delta(\mathbf{r})\delta(t) \tag{1.3}$$

to the right-hand side of Eq. (1.1). The parameter Q_0 has the meaning of the total amount of the substance in the space at the initial instant of time. Despite the very special form of the initial conditions (1.2), the investigation of problem (1.1), (1.2) is of wider interest, since these conditions introduce a new dimensional parameter Q_0 into the treatment, which can turn out to be important. It was assumed in earlier investigations that the asymptotic form must not depend on this parameter and that it is solely defined by the parameters occurring in the equation.

When $\lambda = 0$, the solution of problem (1.1), (1.2) can be expressed in terms of Green's function of the diffusion equation $G(\mathbf{r}, t)$ the relation

$$C^{(0)}(\mathbf{r},t) = Q_0G(\mathbf{r},t), \quad G(\mathbf{r},t) = \frac{\Theta(t)}{[4\pi D_0 t]^{d/2}}\exp\left(-\frac{r^2}{4D_0 t}\right) \tag{1.4}$$

(d is the dimension of the space and $\Theta(t)$ is the Heaviside step function).

When $\delta = 0$, the solution of Eq. (1.1) is given by the relation

$$C(\mathbf{r}, t) = Q_0\exp(-\lambda t)G(\mathbf{r}, t) \tag{1.5}$$

which corresponds to an exponential decrease in the total amount of the substance in the space.

When $\delta \neq 0$, the solution cannot be obtained in general form and the aim of this paper will be to construct an approximate solution of the Cauchy problem. In the case of small values of δ, it can be assumed that the non-linear effects will be small, and it is natural to seek a solution in the form of an expansion in δ while, in the case of arbitrary values of δ, an analytic extension with respect to δ to the specified value can be used. In a certain sense, this procedure is analogous to the method of ε-expansion, which has been successfully used (but without sufficient substantiation) in the theory of critical phenomena, and to the method of dimensional regularization in quantum field theory [7].

For small λ the solution could be sought using perturbation theory [8]. However, in this case, the perturbation is singular since, when $\lambda = 0$, the equation admits of a group with a symmetry of the scale transformation type $\mathbf{r} \to \alpha\mathbf{r}, t \to \alpha^2 t$ and the non-linear term violates the above-mentioned scale invariance. The singular character of the perturbation is reflected in the fact that the series does not converge uniformly in perturbation theory which, in the case of an expansion in a set of eigenfunctions of the unperturbed problem, manifests itself in the occurrence of secular terms in the case of a discrete eigenvalue spectrum and the divergence of the expansion coefficients in the case of a continuous spectrum. In particular, this is seen from formula (1.5), which should be regarded as the result of the summation of the infinite series of perturbation theory in powers of λ (actually, in powers of λt), which does not satisfy the criterion for uniform convergence and is considered as an asymptotic series.

The renormalization group (RG) method, which arose for the first time in quantum field theory [9] and was subsequently successfully applied in the theory of critical phenomena in the case of phase transitions of the second kind [5, 6, 10], is very convenient for summing the infinite series of perturbation theory and is overcoming the difficulties which arise due to the presence of secular terms. Two somewhat differing formulations of the RG method exist. In the so-called field formulation, the RG method enables one, on the basis of a knowledge of the first term of the series in perturbation theory, to predict the structure of the subsequent terms of the series and to sum a certain infinite subsequence of the complete series starting from the existence of a certain arbitrariness in the subdivision of the system being investigated into an unperturbed part and the perturbation (renormalization invariance) [9]. Within the framework of the other (the Wilson) formulation, a system of many interacting modes, which corresponds to the non-linear equation, is considered and equations for the low-frequency (slow) modes are obtained as a result of successive iterative averaging over the high-frequency (fast) modes [10]. When the frequencies of the slow and fast modes are separated, the idea behind this method is equivalent to the Krylov–Bogolyubov method of averaging [11], which is used in the theory of non-linear vibrations. In a substantially multimode system, when the modes of all scales are equally important in understanding the behaviour of the whole system, the property of renormalization invariance implies that the behaviour of the system in the asymptotic domain of long times is independent of the method of separation of the spectrum into slow and fast parts.

Only the first (field) approach is used below to investigate problem (1.1), (1.2) and a detailed explanation of all of the successive stages in the use of the RG method is given. It is assumed here that the reader does not have any prior knowledge of the ideas and techniques of the RG method.

2. THE CONSTRUCTION OF A RENORMALIZED PERTURBATION THEORY

We will change from differential equation (1.1) with initial condition (1.2) to the integral equation

$$C(\mathbf{r},t)=Q_0G(\mathbf{r},t)-\lambda\int_0^t dt'\int d\mathbf{r}'G(\mathbf{r}-\mathbf{r}',t-t')C^{1+2\delta}(\mathbf{r}',t') \tag{2.1}$$

Successive iteration of this equation leads to a representation of the solution in the form of a series in powers of the non-linearity parameter λ (actually in powers of $\lambda Q_0^{1+2\delta}$). The solution of the linear problem is used as the zeroth approximation here. In order to construct a renormalized perturbation theory we renormalize the parameter Q_0, which involves making the substitution $Q_0 \to Q = ZQ_0$ in (2.1), where Z is the renormalization constant. In order to compensate for the effect of renormalization, we add a so-called counter term of the form $(Q_0-Q)G(\mathbf{r},t) = (Z^{-1}-1)QG(\mathbf{r},t)$ to the right-hand side of (2.1) and we treat the non-linear term plus the counter term as the perturbation. As a result of successive iteration, a renormalized perturbation theory series in powers of the parameter $\lambda Q^{1+2\delta}$ is obtained. However, the choice of the renormalization constant Z (and, thereby, the parameter Q) is ambiguous, that is, the subdivision of the right-hand side of (2.1) into a perturbed part and a perturbation is non-unique. While each term of the perturbation theory series will depend on the choice of Z, the complete series must be independent of the choice of the renormalization constant, that is, the complete series must possess renormalization invariance. The requirement that the complete perturbation theory series is renormalization invariant leads to the existence of a certain link between the different terms in the series, and it thereby becomes possible, from a knowledge of the lower approximations of perturbation theory, to find the subsequent terms of the series without recourse to an iterative procedure and a calculation of the higher approximations.

In the first approximation of perturbation theory, the solution can be represented in the form

$$C^{(1)}(\mathbf{r},t)=QG(\mathbf{r},t)-\lambda Q^{1+2\delta}\int_0^t dt'\int d\mathbf{r}'G(\mathbf{r}-\mathbf{r}',t-t')G^{1+2\delta}(\mathbf{r}',t')+(Z^{-1}-1)QG(\mathbf{r},t) \tag{2.2}$$

Using the actual form of Green's function for the diffusion equation (the second formula of (1.4)). It can be shown that the relations

$$G^{1+2\delta}(\mathbf{r},t)=\frac{A}{(D_0t)^{\delta d}}G\left(\mathbf{r},\frac{t}{1+2\delta}\right),\quad A=\frac{1}{(1+2\delta)^{d/2}(4\pi)^{\delta d}} \tag{2.3}$$

hold.

Green's function of the diffusion equation satisfies the so-called half-group law which, when applied to Markov random processes is usually known as the Einstein–Kolmogorov equation for conditional probability [12]

$$C(\mathbf{r},t-t_0)=\int G(\mathbf{r}-\mathbf{r}',t-t')G(\mathbf{r}',t'-t_0)d\mathbf{r}'\quad (t\geqslant t'\geqslant t_0) \tag{2.4}$$

Using (2.3) and (2.4), we integrate over $\mathbf{r}'$ in Eq. (2.2) and find

$$C^{(1)}(\mathbf{r},t)=QG(\mathbf{r},t)-\frac{\lambda Q^{1+2\delta}}{D_0^{\delta d}}A\int_0^t\frac{1}{t'^{\delta d}}G\left(\mathbf{r},t-\frac{2\delta}{1+2\delta}t'\right)dt'+(Z^{-1}-1)QG(\mathbf{r},t) \tag{2.5}$$

The dependence of Green's function on t' under the integral sign on the right-hand side of (2.5) can be neglected for small δ. For sufficiently large δ ($\delta d \geqslant 1$), the domain $t' \approx 0$ makes the main contribution to the integral over t' and it again becomes possible to neglect the dependence of G on t' under the integral sign. In this approximation

$$C^{(1)}(\mathbf{r},t) \equiv q^{(1)}(t)G(\mathbf{r},t), \quad q^{(1)}(t) = Q\left[1 - \frac{\lambda Q^{2\delta}}{D_0^{\delta d}} A I_{\delta d}(0,t)\right]$$

$$I_k(a,b) = \int_a^b \frac{d\eta}{\eta^k} \tag{2.6}$$

that is, the solution has the form of a diffusively spreading density distribution of a substance of which there is a variable total amount $q(t)$.

In formula (2.6), the renormalization constant Z and the quantity associated with it $Q = ZQ_0$ are arbitrary. We now need to satisfy the normalization condition, according to which the amount of the substance at the instant of time $t = \tau$ must be equated to Q, that is, $q(\tau) = Q$. We find the renormalized constant Z from this condition and obtain

$$q(t) \approx Q\left[1 - \frac{\lambda Q^{2\delta}}{D_0^{\delta d}} A I_{\delta d}(\tau,t)\right] = Q\left[1 - \frac{\lambda Q^{2\delta}\tau^{1-\delta d}}{D_0^{\delta d}} A I_{\delta d}(1,t/\tau)\right] \tag{2.7}$$

As a result, the arbitrariness in the choice of the renormalization constant Z is replaced by an arbitrariness in the choice of the normalization point τ.

3. RENORMALIZATION GROUP INVARIANCE AND THE RENORMALIZATION GROUP METHOD

The time-dependence of the amount of substance is a certain function of the parameters λ, D_0, Q, τ and, in this case, due to the arbitrariness in the choice of τ, the form of the function must not be changed on changing from one set of parameters τ, Q to another set τ_1, Q_1, which also reflects the property of renormalization invariance. In a wider sense, this property has been given the name of functional self-similarity [10], which generalizes the concept of self-modelling (self-similarity), which is associated with arbitrariness solely in the choice of scales. The universality of the geometry of hydrodynamic flows, which depend solely on the Reynolds number $\mathrm{Re} = lu/\nu$ but not on the characteristic scales of length, l, velocity, u, and the coefficient of kinematic viscosity, ν is the simplest example of self-similarity.

On the basis of dimensional considerations and the requirement of renormalization invariance, it is possible to write

$$q(t) = Qf(x,g) = Q_1 f(x_1,g_1)$$

$$x = \frac{t}{\tau}, \quad g = \frac{\lambda Q^{2\delta}\tau^{1-\delta d}}{D_0^{\delta d}}, \quad x_1 = \frac{t}{\tau_1}, \quad g_1 = \frac{\lambda Q_1^{2\delta}\tau_1^{1-\delta d}}{D_0^{\delta d}}, \quad Q_1 = Qf\left(\frac{\tau_1}{\tau}, g\right) \tag{3.1}$$

The last equality of (3.1) follows from the relation $f(1,q) = 1$ which, in turn, follows from the normalization condition $q(\tau) = Q$.

We now introduce the new dimensionless function

$$\tilde{g} = \lambda q(t)^{2\delta} t^{1-\delta d} / D_0^{\delta d} \equiv g f^{2\delta}(x,g) x^{1-\delta d} \tag{3.2}$$

This function turns out to be an invariant of the transformation $\tau \to \tau_1$, $Q \to Q_1$, that is

$$\tilde{g}(x,g) = \tilde{g}(x_1,g_1) \tag{3.3}$$

The function $\tilde{g}(x, g)$ is a dimensionless, time-dependent, real parameter of the expression in a perturbation theory series. By virtue of the condition $f(1,g) = 1$, this function obeys the normalization condition

$$\tilde{g}(1,g)=g \tag{3.4}$$

and satisfies the functional equation

$$\tilde{g}(x,g)=\tilde{g}(x/\alpha,\ \tilde{g}(\alpha,g)),\quad \alpha=\tau_1/\tau \tag{3.5}$$

It follows from (3.4) and (3.5) that the set of transformations $(\tau, g) \to (\tau_1, g_1)$ satisfies the group composition law $(\tau, g) \to (\tau_2, g_2) = (\tau, g) \to (\tau_1, g_1) \to (\tau_2, g_2)$ it has identity and inverse elements and thereby forms a continuous single parameter group which is called the renormalization group (RG).

On differentiating (3.5) with respect to α and then putting $\alpha = 1$, we find the differential equation of the RG

$$\left\{-x\frac{\partial}{\partial x}+\beta(g)\frac{\partial}{\partial g}\right\}\tilde{g}(x,g)=0,\quad \beta(g)=\frac{\partial\tilde{g}(x,g)}{\partial x}\Big|_{x=1} \tag{3.6}$$

Equation (3.6) is too general; all the actual information concerning the system under consideration is contained in the so-called renormalization group function (the RG-function) $\beta(g)$ which is determined, according to (3.6), by the behaviour of the function $\tilde{g}(x, g)$ close to the normalization point $x = 1$. The RG-function, $\beta(g)$, plays the role of an infinitesimal transformation operator (a generator) of the renormalization group.

The RG method consists of the proposal to use the renormalized perturbation theory to calculate the function $\beta(g)$ [9]. If the function β is calculated in the lowest approximation of perturbation theory and substituted into Eq. (3.6), then the subsequent solution of this equation will correspond to the summation of a certain infinite subsequence of the full perturbation theory series.

The following illustrative example can be given for clarification. If it is known that a certain quantity is the sum of a geometric progression, then a knowledge of the first two terms of the sum enables one to find all the remaining terms and the sum. The requirement of renormalization group invariance, which is expressed by the RG equations (3.5) and (3.6), is the analogue, in the renormalization group approach, of the knowledge of the fact that the treatment refers to a geometrical progression.

4. SOLUTION OF THE RENORMALIZATION GROUP EQUATIONS

It follows from (2.7) that

$$f(x,g)\approx 1-AgI_{\delta d}(1,x) \tag{4.1}$$

Using (3.6) and (4.1), we find

$$\beta(g)=-2\delta Ag(g-g^*),\quad g^*=(1-\delta d)/(2\delta A) \tag{4.2}$$

Problem (3.6) for case (4.2) will be solved later. Now we shall discuss the asymptotic form of the solution in the case of an RG-function of general form. According to the methods of classical mechanics, the asymptotic solution of equations of the type (3.6) is given by the fixed points g_i, which satisfy the condition $\beta(g_i) = 0$. Here, a fixed point is stable (attracting) in the domain of large values of $x = t/\tau$ subject to the condition $\partial\beta(g)/\partial g|_{g=g_i} < 0$, and unstable (repelling) otherwise. If there is an asymptotically stable fixed point, then, according to (3.2), the asymptotic form of the function $f(x, g)$ will have the form

$$f(x,g)\sim x^{-(1-\delta d)/(2\delta)} \tag{4.3}$$

It follows from (4.2) that there are two fixed points: the trivial point $g = 0$, which corresponds to the fact that there are no non-linear interactions (asymptotic freedom), and the non-trivial point $g = g^*$ when the non-linear term turns out to be substantial. According to (4.2), the non-trivial fixed point g^* will be stable in the domain of long times subject to the condition $\delta d < 1$ and the trivial fixed point is found to be stable when $\delta d > 1$. When $\delta d = 1$, the trivial and non-trivial fixed points merge and, in order to determine the asymptotic form, it is necessary to seek a new stable fixed point by calculating the renormalization group function (the RG-function) in higher approximations of perturbation theory. The value of the dimension $d_c = 1/\delta$ at which there is a crossover in the stability of the fixed points is called the crossover dimension.

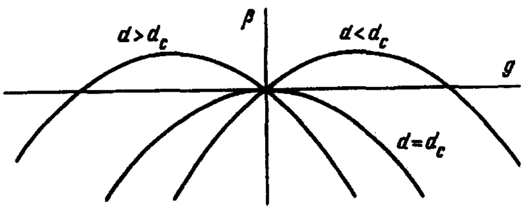

Fig. 1.

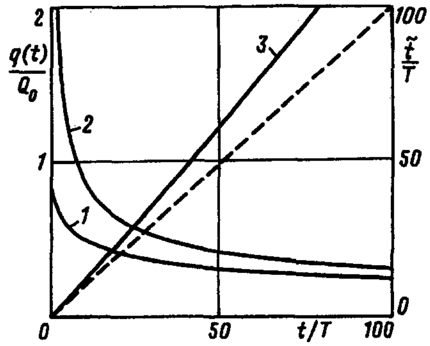

Fig. 2.

For binary-type reactions ($\delta = 1/2$), the crossover dimension will correspond to the planar case ($d_c = 2$). Hence, in the case of binary-type reactions, a non-trivial state of asymptotic behaviour will only occur in a one-dimensional problem, the two-dimensional case requires special consideration and the absence of a non-linear interaction (asymptotic freedom) will correspond to the three-dimensional case.

The form of the RG-function for three types of asymptotic behaviour of the solution is shown in Fig. 1.

In order to investigate when the asymptotic state is reached, we will find the exact solution of Eq. (3.6), which corresponds to an RG-function of the form (4.2). This solution is found by the method of characteristics and is given in implicit form by the Gell-Mann–Low formula [9]

$$\int_{g}^{\bar{g}(x,g)} \frac{dg'}{\beta(g')} = \ln x \tag{4.4}$$

On substituting (4.2) into (4.4), integrating with respect to g' and solving Eq. (4.4) for the unknown function, we find the function $\bar{g}(x, g)$, a knowledge of which enables us, using (3.2), to obtain

$$f(x,g) = \left[1 + \frac{g}{g^*}(x^{1-\delta d} - 1)\right]^{-\xi}, \quad \xi = \frac{1}{2\delta} \tag{4.5}$$

On returning to the initial (unrenormalized) parameters and introducing the characteristic time T by the relation $\lambda Q_0^{2\delta}/D_0^{\delta d} = T^{\delta d-1}$, we find

$$q(t) = Q_0\left[1 + \frac{1}{g^*}\left(\frac{t}{T}\right)^{1-\delta d}\right]^{-\xi} \tag{4.6}$$

Result (1.5), which corresponds to the exact solution of the problem, can be obtained in the limiting case when $\delta \to 0$ using the relation $[q(t)/\Omega_0]^{2\delta} \simeq 1 + 2\delta \ln[q(t)/Q_0]$. When $\delta = 0$, application of the RG-method actually reproduces the method of variation of a constant which is well known in the theory of linear differential equations.

The dependence of the amount of substance on the dimensionless time, t/T, corresponding to formula (4.6), is shown in Fig. 2 (curve 1) for the case of a one-dimensional diffusion process ($d = 1$) when there are binary-type reactions ($\delta = 1/2$). For comparison, we also show curve 2, which corresponds to the asymptotic power solution. It can be seen that an asymptotic state is reached quite slowly in this case.

5. REFINEMENT OF THE APPROXIMATION OF SMALL δ

It was found that in the formula of the first approximation of renormalization perturbation theory (2.5), the dependence of Green's function in the integrand on t' should be neglected. As a result, the effect of non-linearity was reduced to the appearance of a time-dependence of the amplitude factor associated with the amount of substance in the system and to the invariability of the pattern for the evolution of the spatial density distribution. It can be shown that taking account of the following term in the expansion with respect to δ in (2.5) reduces to the replacement of the diffusion coefficient by a certain effective, time-dependent diffusion coefficient $\tilde{D}(t)$, which determines the rate of the spreading of the density distribution of the substance.

The preceding treatment, when account is taken of the change in the rate of diffusive spreading due to non-linearity without a detailed exposition of the RG method and when attention is solely directed to the differences which arise, is reproduced below within the framework of the renormalization group approach. These differences mainly lie in the fact that renormalization of the initial conditions was carried out in the case treated above and that the diffusion coefficient in the initial differential equation is also renormalized in the given treatment.

We now renormalize the diffusion coefficient in the initial equation (1.1) by means of the substitution $D_0 \to D = Z_1 D_0$ and add a counter term which compensates for this change to the right-hand side. After changing from the differential equation to the integral equation and subsequent renormalization of the parameter Q, we obtain an equation which differs from (2.2) in that there is an additional term on the right-hand side which is equal to

$$(Z_1^{-1}-1)D\Delta\int_0^t dt'\int d\mathbf{r}'\, G(\mathbf{r}-\mathbf{r}',t-t')C(\mathbf{r}',t')$$

Here, Green's function is constructed using the renormalized value of D rather than D_0 as in Section 2.

On carrying out similar calculations in the lowest non-vanishing approximation of perturbation theory while taking account of the next term in the expansion in δ of the argument of Green's function in the integrand with respect to t' and using the equations for the renormalization of Green's function, we find

$$C^{(1)}(\mathbf{r},t)=\left\{Q-\frac{\lambda Q^{1+2\delta}}{D^{\delta d}}AI_{\delta d}(0,t)+(Z^{-1}-1)Q\right\}G(\mathbf{r},t)+$$
$$+\left\{-\frac{\lambda Q^{1+2\delta}}{D^{\delta d}}\frac{2\delta D}{(1+2\delta)^{\delta d}}AI_{\delta d-1}(0,t)+(Z_1^{-1}-1)QDt\right\}\Delta G(\mathbf{r},t) \tag{5.1}$$

The terms which are proportional to G are treated in a similar manner.

To determine the reormalization constant of the diffusion coefficient Z_1 we require that, when $t = \tau$, the correction to the renormalized diffusion coefficient, which is proportional to ΔG, should vanish. As a result, in the lowest approximation of perturbation theory, we find

$$\frac{1}{Z_1(\tau)}=\frac{D_0}{D}=1+\frac{\lambda Q^{2\delta}}{D^{\delta d}}\frac{2\delta}{1+2\delta}\frac{A}{\tau}I_{\delta d-1}(0,\tau)=1+2aAgI_{\delta d}(0,\tau)$$
$$2a=\frac{2\delta}{1+2\delta}\frac{1-\delta d}{2-\delta d} \tag{5.2}$$

The requirement that the corrections to the diffusion coefficient should vanish corresponds to the normalization condition $\tilde{D}(t)|_{t=\tau} = D$, from which it follows that

$$\tilde{D}(t)=D_0Z_1(t)=DZ_1(t)Z_1^{-1}(\tau)=Df_1(t/\tau,\lambda Q^{2\delta}\tau^{1-2\delta}/D^{\delta d}) \tag{5.3}$$

By (5.2), the function $f_1(x, g)$ in the lowest approximation of perturbation theory has the form

$$f_1(x,g)=1+2aAgI_{\delta d}(1,x),\quad g=\frac{\lambda Q^{2\delta}\tau^{1-\delta d}}{D^{\delta d}} \tag{5.4}$$

By analogy with (3.2), we introduce the function

$$\tilde{g}(x,g)=\frac{\lambda q^{2\delta}(t)t^{1-\delta d}}{\tilde{D}^{\delta d}(t)}=g\frac{f^{2\delta}(x,g)}{f_1^{\delta d}(x,g)}x^{1-\delta d} \tag{5.5}$$

which is an invariant of the renormalization group transformation $\tau \to \tau_1$, $Q \to Q_1$, $D \to D_1$.

The quantity $\tilde{g}(x, g)$ is calculated in the same way as in Section 2; the sole difference lies in the substitution $A \to A^* = A(1 + ad)$ in the corresponding formulae. However, the calculation of the functions $f(x, g)$ and $f(x, g)$ now turns out to be not so trivial and it is now necessary additionally to solve the equations separately for f and f_1.

It follows from (3.1) that the function $f(x, g)$ (as well as the function $f_1(x, g)$ satisfies the RG functional equation

$$f(x, g) = f(\alpha, g) f(x/\alpha, \tilde{g}(\alpha, g)) \tag{5.6}$$

Differentiating Eq. (5.6) with respect to α and then putting $\alpha = 1$, we obtain the RG differential equation for $f(x, g)$ (the same equation is obtained for $f_1(x, g)$)

$$\left\{-x\frac{\partial}{\partial x} + \beta(g)\frac{\partial}{\partial g}\right\}\ln f(x,g) = -\gamma(g), \quad \gamma(g) = \left.\frac{\partial f(x,g)}{\partial x}\right|_{x=1} \tag{5.7}$$

The general solution of the linear equation (5.7) can be represented in the form of a sum of the particular solution of the inhomogeneous equation $\Phi(g)$ and the general solution of the homogeneous equation, which is an arbitrary function of the characteristic $F(\tilde{g}(x, g))$

$$\ln f(x, g) = F(\tilde{g}(x, g)) + \Phi(g) \tag{5.8}$$

Putting $x = 1$ in (5.8) and making use of the normalization conditions (3.1) and (3.4), we find

$$F(g) = -\Phi(g) \tag{5.9}$$

whence it follows that

$$f(x,g) = \frac{\exp\{-\Phi(\tilde{g}(x,g))\}}{\exp\{-\Phi(g)\}}, \quad \Phi(g) = -\int \frac{\gamma(g)dg}{\beta(g)} \tag{5.10}$$

On calculating $\gamma(g)$ and $\gamma_1(g)$ from (4.1) and (5.4), using (5.9) and (5.10) we obtain formulae analogous to (4.5).

Returning from the renormalized values of Q and D to the initial parameters Q_0, D_0 and using the characteristic time T, we find formulae for $f(x, g)$ and $f_1(x, g)$, analogous to (4.6), in which

$$g^* = \frac{1-\delta d}{2\delta(1+ad)}, \quad \xi = \frac{1}{2\delta(1+ad)}, \quad \xi_1 = -\frac{a}{\delta(1+ad)} \tag{5.11}$$

Note that formula (4.5), which was obtained without taking account of the renormalization of the diffusion coefficient, is reproduced when $a = 0$.

If an effective time $\tilde{t}$ is now introduced, using the relation $\tilde{D}(t)\,t = D_0\tilde{t}$, then, in terms of this effective time, the solution can be represented in the form

$$C(\mathbf{r}, t) = Q_0(\tilde{t}/t)^{-1/(2a)} G(\mathbf{r}, \tilde{t}) \tag{5.12}$$

The relation between the dimensionless effective time $\tilde{t}/T$ and the dimensionless time t/T is as follows:

$$\frac{\tilde{t}}{T} = \frac{t}{T}\left[1 + \frac{1}{g^*}\left(\frac{t}{T}\right)^{1-\delta d}\right]^{-\xi_1} \tag{5.13}$$

The corresponding curve for the case when $\delta = 1/2$ and $d = 1$ is shown in Fig. 2 (curve 3) and, for comparison, the dependence $\tilde{t} = t$ is represented by the dashed line.

Since, according to (5.13), $\tilde{t} > t$ when $d < d_c$, this means that taking account of non-linear effects leads to an increase in the rate of spreading of the initially localized density distribution and the radius of the domain of localization increases as $r \sim t^{1/2+\alpha}$, where

$$\alpha = \frac{1-\delta d}{2\delta}\frac{a}{1+ad} \tag{5.14}$$

Note that the exponent α is universal, that is, it is independent of the values of the characteristic parameters of the problem Q_0, D_0, λ and is determined solely by the scale d and the order of the reaction δ.

6. DISCUSSION

The principal aim of this paper is to demonstrate the possibility of using a renormalization group approach in problem (1.1), (1.2). However, in order to reveal the possibilities and means of applying the RG method to other problems in mathematical physics it is useful to reproduce the logical basis of the method employed above, which has not always been seen against the background of the formal calculations, and to direct attention to the non-trivial results obtained above using the RG method.

At first glance it may appear that, due to diffusive spreading of the initially localized density distribution of a substance and the decrease in concentration as a consequence of non-linear attenuation at long times, a state of asymptotic freedom must be reached for which the non-linear term in Eq. (1.1) becomes negligibly small compared with the diffusion term, the evolution of the concentration distribution will be solely determined by diffusion processes, and the asymptotic form will turn out to be self-similar. However, the above analysis showed that a self-similar state of asymptotic freedom is only attained in the case when the space dimension exceeds a certain critical value $d_c = 1/\delta$. The result which has been obtained leads to the conclusion that a non-trivial state is reached when $d < d_c$ for which a certain stable balance is established between the diffusive spreading processes and non-linear attenuation. The evolution of the spatial distribution is found to be self-similar, that is, scale similarity (scaling) holds at different times. According to the terminology adopted, this self-similarity is incomplete [2] since the exponential is not determined by dimension considerations. The value of the incomplete self-similarity factor which has been found turns out to be universal, that is, it is independent of the characteristic parameters of the problem and, in particular, the non-linear interaction constant λ. This is indicated by the fact that the dependence of the solution on λ is not analytic at the point $\lambda = 0$ which also confirms the singular nature of the perturbations, associated with the breakdown of the above-mentioned symmetry properties of the unperturbed equation.

The question of the degree of accuracy and the domain of applicability of the solution which has been obtained as well as the connection between the RG method and perturbation theory is important. On the one hand, the use of an iterative procedure when constructing the solution means that a sequence of perturbation theories is sought depending on the magnitude of the non-linear interaction constant. However, the formal expansion parameter $\lambda Q_0^{2\delta}/D_0^{2\delta}$, which emerges in this case, has a non-zero dimension, and the quantity $\lambda Q_0^{2\delta} t^{1-\delta d}/D_0^{\delta d}$, which is time-dependent, is the actual expansion parameter. Although the renormalization procedure $Q_0 \to Q$, $D_0 \to D$ also leads to a state of affairs where the actual expansion parameter, constructed using the renormalized values of the parameters of the problems, is found to be smaller than the parameter constructed using initial (unrenormalized) values, this only leads to an improvement in the rate of convergence of the series in the case of a specified final time. However, the determination of the asymptotic form for long periods of time remains outside the scope of standard perturbation theory.

The property of renormalized invariance, which establishes a definite link between the different terms of a perturbation theory series, turns out to be extremely useful in determining the asymptotic form. As a result of renormalization invariance, the possibility arises of finding these terms and summing the whole series (or an infinite subsequence of this series) without making any assumptions regarding the smallness of the expansion parameter.

In this case, the possibility of carrying out a renormalization procedure (renormalizability) means that the effect of non-linear perturbations does not change the overall density distribution pattern and only leads to the total amount of the substance, and the diffusion coefficient becoming time-dependent. As a result, the search for the solution reduces to the determination of the corresponding dependences. The time-dependent quantities $q(t)$ and $\bar{D}(t)$ differ substantially from the initial constant values of Q_0 and D_0, and this is a formal indication of the non-applicability of perturbation theory. Renormalization means the replacement of the initial values by renormalized values, which are selected in such a way that the renormalized values are identical with the effective values at the point of normalization $t = \tau$. According to the renormalizability hypothesis, close to the point of normalization the solution will have the form of the distribution when there are no non-linear interactions but with the renormalized values of the parameters Q and D. The large corrections which thereby arise in the zeroth approximation due to renormalization are found to be included in the zeroth approximation of renormalized perturbation theory, which can give reasonable results in a bounded domain and can be used to calculate the RG function which is determined by the behaviour of the solution at the point of normalization. Unlike in quantum field theory, where renormalizability is associated with the possibility of removing divergences, in the problem under consideration renormalizability may turn out to be approximate and is only valid in a certain asymptotic domain. It is seem from the calculations which have been carried out (formulae (2.6) and (5.2)) that renormalization is found to be possible in the approximation of small values of the exponent δ. However, this does not mean that a perturbation theory is constructed with respect to δ since the solution for the case when $\delta = 0$ is not used as the zeroth approximation. Moreover, it is found that the exactly solvable case when $\delta = 0$ is obtained in the limit when $\delta \to 0$ by the RG method using the lowest approximation of perturbation theory with respect to λ when calculating the RG function.

As δ increases, the property of renormalizability, which, in the case under consideration, reduces to the possibility of neglecting the dependence on t' in Green's function in the integrand in formula (2.5) and the formula of Section 5 corresponding to it, which has not been written out, becomes ever more approximate. However, when δ is increased further, when $\delta \geqslant 1/d$, the integrals over t' become singular at the point $t' = 0$. In this case, the possibility arises of replacing Green's function by its value at the singular point and taking it outside the integral sign (formulae (2.6) and (5.1)). Such a procedure has been used in the analysis, using the RG method, of a certain version of a non-linear generalization of the diffusion equation and a value of the partial self-similarity exponent was obtained which is in good agreement with the results of a numerical solution [14, 15]. Hence, the use of the formulae which have been obtained in the case of arbitrary δ is an extrapolation of the property of renormalizability which holds when $\delta \ll 1$ and when $\delta \geqslant 1/d$ in the domain of arbitrary δ. This approach is similar to the method of scale regularization [4, 14] used in quantum field theory, according to which integrals which diverge in the case of the physical dimension of the space are replaced by integrals obtained in the case of a analytic extension of the dimension of the space of the corresponding integrals for a dimension when these integrals are well defined (finite). In order to illustrate this idea, we point to the fact that the gamma-function for negative values of the argument is not defined by means of an Euler integral (which is divergent) but using the recurrence formula $\Gamma(x + 1) = x\Gamma(x)$ [16].

Note that, in the case of a dimension close to the critical value, that is, when $d = d_c + \varepsilon(\varepsilon \to 0)$, the actual expansion parameter in the perturbation theory series in the asymptotic domain g^* is found to be proportional to ε. The ε-expansion procedure, which has been successfully used in the theory of critical phenomena, reduces to the construction of a perturbation theory in powers of ε with a subsequent analytic extension with respect to ε to a point corresponding to the dimension of real space [5, 6]. The analogy with the theory of critical phenomena can serve as some substantiation for the approximations used in obtaining formulae (2.6) and (5.2).

In conclusion, we point out that an analogous treatment of transport phenomena, which are accompanied by the production of a substance ($\lambda < 0$), leads to the fact that a non-trivial fixed point will be stable when $d > d_c$ and a state of asymptotic freedom is attained when $d < d_c$. The effective time will decrease as the real time increases, which denotes a tendency to localization of initially weakly localized distributions and an increase in their slope (states with peaking [1]).

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